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Higher-order self-adjoint boundary-value problems on time scales

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Abstract

In this study, higher-order self-adjoint differential expressions on time scales and their associated self-adjoint boundary conditions are discussed. The symmetry property of the corresponding Green's functions is shown, together with specific formulas of Green's functions for select time scales.

MSC: 34B05; 39A10

Keywords: Time scales; Self-adjoint boundary-value problem; Symmetric Green's function

1. Introduction

Dynamic equations on time scales have been introduced [21] to unify and extend the theory of ordinary differential equations, difference equations, quantum equations (based on the q-calculus and the h-calculus) [25], and all other differential systems defined over non-empty closed subsets of the real line. Already several important problems concerning higher-order dynamic equations on time scales, involving only delta differentiation, have been developed [3,12–14,17,20,23,24]. In [9], self-adjoint boundary-value problems (BVPs) for second-order dynamic equations on time scales were introduced and examined by making use of both delta and nabla derivatives. Next some BVPs for higher-order equations on time

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scales involving alternating or stacked delta and nabla derivatives were investigated in [4–8], where the considered BVPs turned out, in general, to be non-self-adjoint because their Green's functions were found non-symmetric. Quite recently one of the authors [19] suggested two classes of higher-order dynamic equations on time scales and associated with them boundary conditions, involving both delta and nabla derivatives simultaneously, which generate self-adjoint BVPs in the classical sense, and hence symmetric Green's functions. These classes of equations can be formulated as follows.

Let $\mathbb{T}$ be a time scale, $p_0, p_1, \ldots, p_n$ real-valued right-dense continuous functions defined on $\mathbb{T}$ with $p_0(t) \neq 0$ for all $t \in \mathbb{T}$, and $a \in \mathbb{T}^{\kappa^n}$, $b \in \mathbb{T}_{\kappa^n}$, with $a < b$. For notation's sake, by $f^{\nabla^{-1}\Delta}$ and $f^{\Delta^{-1}\nabla}$ we mean the function *f*.

Then any $2n$th-order differential expression

$$Ly(t) = \sum_{i=0}^{n} (-1)^{n-i} (p_i y^{\Delta^{n-i-1}\nabla})^{\nabla^{n-i-1}\Delta}(t) = (-1)^n (p_0 y^{\Delta^{n-1}\nabla})^{\nabla^{n-1}\Delta}(t) + \cdots$$
$$- (p_{n-3} y^{\Delta^2\nabla})^{\nabla^2\Delta}(t) + (p_{n-2} y^{\Delta\nabla})^{\nabla\Delta}(t) - (p_{n-1} y^{\nabla})^{\Delta}(t) + p_n(t) y(t) \tag{1.1}$$

is formally self-adjoint with respect to the inner product

$$\langle y, z \rangle = \int_a^b y(t) z(t) \Delta t,$$

that is, the identity

$$\langle Ly, z \rangle = \langle y, Lz \rangle$$

holds provided that y and z satisfy some appropriate self-adjoint boundary conditions at a and b.

Similarly, the differential expression

$$My(t) = \sum_{i=0}^{n} (-1)^{n-i} (p_i y^{\nabla^{n-i-1}\Delta})^{\Delta^{n-i-1}\nabla}(t) = (-1)^n (p_0 y^{\nabla^{n-1}\Delta})^{\Delta^{n-1}\nabla}(t) + \cdots$$
$$- (p_{n-3} y^{\nabla^2\Delta})^{\Delta^2\nabla}(t) + (p_{n-2} y^{\nabla\Delta})^{\Delta\nabla}(t) - (p_{n-1} y^{\Delta})^{\nabla}(t) + p_n(t) y(t) \tag{1.2}$$

is formally self-adjoint with respect to the inner product

$$\langle y, z \rangle = \int_a^b y(t) z(t) \nabla t.$$

In the present paper, we give a detailed presentation for the dynamic expression (1.1). The paper is organized as follows.

In Section 2, some time scale essentials are included for the convenience of the reader. In Section 3 we consider the dynamic expression (1.1). Here, the quasi-derivatives of a function are introduced in terms of which the equation $Ly = g$ is written as an equivalent first-order system, and using this first-order system an existence and uniqueness theorem for solutions of $Ly = g$ is presented. In Section 4, a definition of boundary conditions which are self-adjoint with respect to the dynamic expression (1.1) is given and the symmetry property of the corresponding Green's functions is emphasized. In Section 5, it is shown that the linear dynamic equation $Ly = 0$ can be written as an equivalent Hamiltonian system. Section 6 details the second-order case, where a general form of self-adjoint boundary conditions is given, and for the case of separated boundary conditions (Sturm–Liouville boundary conditions) the corresponding Green's function is constructed. Section 7 develops the fourth-order case. Here, several examples of self-adjoint boundary conditions and corresponding Green's functions are presented. Section 8 briefly describes the dynamic expression (1.2). Finally, Section 9 discusses additional adjoint and self-adjoint forms, namely those with alternating derivatives and those with stacked derivatives.

2. Time scale essentials

Any arbitrary non-empty closed subset of the reals $\mathbb{R}$ can serve as a time scale $\mathbb{T}$; see [10,13,14,21].

Definition 2.1. For $t \in \mathbb{T}$ define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

It is convenient to have the graininess operators $\mu_\sigma : \mathbb{T} \to [0, \infty)$ and $\mu_\rho : \mathbb{T} \to (-\infty, 0]$ defined by $\mu_\sigma(t) = \sigma(t) - t$ and $\mu_\rho(t) = \rho(t) - t$, respectively. In the case where $\mathbb{T}$ has a maximum point M, we define $\sigma(M) = M$. If $\mathbb{T}$ has a minimum point m, define $\rho(m) = m$. A point $t \in \mathbb{T}$ is left-scattered if $\mu_\rho(t) \neq 0$ and left-dense if $\mu_\rho(t) = 0$; there are analogous notions for right-scattered and right-dense using μ_σ. Let $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$ if $\mathbb{T}$ has a left-scattered maximum M, and $\mathbb{T}^\kappa = \mathbb{T}$ otherwise. Similarly, $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$ if $\mathbb{T}$ has a right-scattered minimum m, with $\mathbb{T}_\kappa = \mathbb{T}$ otherwise. In addition use the notation $\mathbb{T}^{\kappa^2} = (\mathbb{T}^\kappa)^\kappa$, etc.

Example 2.2. The sets $\mathbb{R}$, $h\mathbb{Z}$ where $h > 0$, and $\mathbb{E} = \{1 - q^{\mathbb{N}_0}\} \cup \{1\}$ where $0 < q < 1$ are examples of time scales. For any $t \in \mathbb{R}$, $\sigma(t) = \rho(t) = t$, and $\mu_\sigma(t) = \mu_\rho(t) = 0$. For any $t \in h\mathbb{Z}$, $\sigma(t) = t + h$, $\rho(t) = t - h$, $\mu_\sigma(t) = h$, and $\mu_\rho(t) = -h$. On the other hand, for $\mathbb{E}$ we have

$$\sigma(t) = 1 - q + qt, \quad \rho(t) = \frac{q - 1 + t}{q},$$

$$\mu_\sigma(t) = (1 - t)(1 - q), \quad \mu_\rho(t) = -\frac{(1 - q)(1 - t)}{q}.$$

We can see that the formula for $\rho(t)$ only holds for $t \neq 1$; when $t = 1$, $\rho(1) = 0$.

Definition 2.3. A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous (rd-continuous) provided it is continuous at all right-dense points of $\mathbb{T}$ and its left-sided limit exists (finite) at left-dense points of $\mathbb{T}$. The set of all right-dense continuous functions on $\mathbb{T}$ is denoted by

$$C_{\mathrm{rd}} = C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}).$$

Similarly, a function $f : \mathbb{T} \to \mathbb{R}$ is left-dense continuous (ld-continuous) provided it is continuous at all left-dense points of $\mathbb{T}$, and its right-sided limit exists (finite) at right-dense points of $\mathbb{T}$. The set of all left-dense continuous functions is denoted

$$C_{\mathrm{ld}} = C_{\mathrm{ld}}(\mathbb{T}) = C_{\mathrm{ld}}(\mathbb{T}, \mathbb{R}).$$

Definition 2.4 (*Delta Derivative*). Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t, such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leqslant \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

The function $f^\Delta(t)$ is the delta derivative of f at t.

Definition 2.5 (*Delta Integral*). Let $f : \mathbb{T} \to \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $F : \mathbb{T} \to \mathbb{R}$, such that $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$, then F is a delta antiderivative of f. In this case the integral is given by the formula

$$\int_a^b f(\tau)\Delta\tau = F(b) - F(a) \quad \text{for } a, b \in \mathbb{T}.$$

Definition 2.6 (*Nabla Derivative*). For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, define $f^\nabla(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t, such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s]| \leqslant \varepsilon|\rho(t) - s| \quad \text{for all } s \in U.$$

The function $f^\nabla(t)$ is the nabla derivative of f at t.

In the case $\mathbb{T} = \mathbb{R}$, $f^\Delta(t) = f'(t) = f^\nabla(t)$. When $\mathbb{T} = \mathbb{Z}$, $f^\Delta(t) = f(t+1) - f(t)$ and $f^\nabla(t) = f(t) - f(t-1)$.

Definition 2.7 (*Nabla Integral*). Let $f : \mathbb{T} \to \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $F : \mathbb{T} \to \mathbb{R}$, such that $F^\nabla(t) = f(t)$ for all $t \in \mathbb{T}_\kappa$, then F is a nabla antiderivative of f. In this case the integral is given by the formula

$$\int_a^b f(\tau)\nabla\tau = F(b) - F(a) \quad \text{for } a, b \in \mathbb{T}.$$

Remark 2.8. All right-dense continuous bounded functions on $[a, b)$ are delta integrable from a to b, and all left-dense continuous bounded functions on $(a, b]$ are nabla integrable from a to b.

Theorem 2.9. *If $f, g : \mathbb{T} \to \mathbb{R}$ and their nabla derivatives f^∇, g^∇ are left-dense continuous then*

$$\int_a^b f(t)g^\nabla(t)\nabla t = (fg)(b) - (fg)(a) - \int_a^b f^\nabla(t)g(\rho(t))\nabla t.$$

Theorem 2.10. *If $f, g : \mathbb{T} \to \mathbb{R}$ and their delta derivatives f^Δ, g^Δ are right-dense continuous then*

$$\int_a^b f(t)g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t.$$

For a more general treatment of the delta and nabla integrals see [14, Chapter 5] and [18]. The following statement [9, Theorems 2.5, 2.6, and 2.10] will play an important role.

Theorem 2.11. (i) *If* $f : \mathbb{T} \to \mathbb{R}$ is Δ-*differentiable on* $\mathbb{T}^\kappa$ *and if* f^Δ *is continuous on* $\mathbb{T}^\kappa$, *then f is* ∇-*differentiable on* $\mathbb{T}_\kappa$ *and*

$$f^\nabla(t) = f^\Delta(\rho(t)) \quad \textit{for all } t \in \mathbb{T}_\kappa.$$

(ii) *If* $f : \mathbb{T} \to \mathbb{R}$ *is* ∇-*differentiable on* $\mathbb{T}_\kappa$ *and if* f^∇ *is continuous on* $\mathbb{T}_\kappa$, *then f is* Δ-*differentiable on* $\mathbb{T}^\kappa$ *and*

$$f^\Delta(t) = f^\nabla(\sigma(t)) \quad \textit{for all } t \in \mathbb{T}^\kappa.$$

For $f : \mathbb{T} \to \mathbb{R}$, we define the functions $f^\sigma : \mathbb{T} \to \mathbb{R}$ and $f^\rho : \mathbb{T} \to \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t)) \quad \text{and} \quad f^\rho(t) = f(\rho(t)) \quad \text{for all } t \in \mathbb{T}.$$

The statements of the previous theorem can be formulated as

$$(f^\Delta)^\rho = f^\nabla \quad \text{and} \quad (f^\nabla)^\sigma = f^\Delta$$

provided that f^Δ and f^∇ are continuous, respectively.

Theorem 2.12. *Let* $f : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ *be a continuous function of two variables* $(t, s) \in \mathbb{T} \times \mathbb{T}$, *and* $a \in \mathbb{T}$. *Assume that f has continuous derivatives* f^Δ *and* f^∇ *with respect to t. Then the following formulas hold*:

(i) $\left(\int_a^t f(t,s)\Delta s\right)^\Delta = f(\sigma(t), t) + \int_a^t f^\Delta(t,s)\Delta s$,

(ii) $\left(\int_a^t f(t,s)\Delta s\right)^\nabla = f(\rho(t), \rho(t)) + \int_a^t f^\nabla(t,s)\Delta s$,

(iii) $\left(\int_a^t f(t,s)\nabla s\right)^\Delta = f(\sigma(t), \sigma(t)) + \int_a^t f^\Delta(t,s)\nabla s$,

(iv) $\left(\int_a^t f(t,s)\nabla s\right)^\nabla = f(\rho(t), t) + \int_a^t f^\nabla(t,s)\nabla s$.

3. Self-adjoint dynamic equations

For the theory of higher-order differential equations refer to [15,26,28,29]. Consider the $2n$th-order dynamic expression (1.1), in which the coefficient functions $p_i : \mathbb{T} \to \mathbb{R}$ are right-dense continuous for $0 \leqslant i \leqslant n$ and $p_0(t) \neq 0$ for all $t \in \mathbb{T}$. Set $\mathbb{T}^* = \mathbb{T}^{\kappa^n}_{\kappa^n} = \mathbb{T}^{\kappa^n} \cap \mathbb{T}_{\kappa^n}$.

Definition 3.1. Let Ω be the linear set of all functions $y : \mathbb{T} \to \mathbb{R}$, such that the function

$$(p_i y^{\Delta^{n-i-1}\nabla})^{\nabla^{n-i-1}\Delta}$$

is defined on $\mathbb{T}^{\kappa^{n-i}}_{\kappa^{n-i}}$ and is right-dense continuous for $0 \leqslant i \leqslant n$.

For each $y \in \Omega$ the expression Ly is defined and presents a right-dense continuous function on $\mathbb{T}^*$.

Remark 3.2. In general, the product of two "smooth" functions on time scales does not have higher-order derivatives; see [13, Example 1.56]. Nevertheless, from the existence and uniqueness theorem given below it follows that the set Ω contains a sufficiency of functions. Indeed, for any right-dense continuous function $g : \mathbb{T} \to \mathbb{R}$, solutions of the equation $Ly(t) = g(t)$ will belong to Ω, and all functions in Ω are obtained in this way.

For each function $y \in \Omega$, at $t \in \mathbb{T}^*$ set

$$y^{[k]} = y^{\Delta^k}, \quad 0 \leqslant k \leqslant n-1, \quad y^{[0]} = y^{\Delta^0} = y, \tag{3.1}$$

$$y^{[n]} = p_0 y^{\Delta^{n-1}\nabla}, \tag{3.2}$$

$$y^{[n+k]} = p_k y^{\Delta^{n-k-1}\nabla} - (y^{[n+k-1]})^{\nabla}, \quad 1 \leqslant k \leqslant n-1, \tag{3.3}$$

$$y^{[2n]} = p_n y - (y^{[2n-1]})^{\Delta}. \tag{3.4}$$

As in the traditional case $\mathbb{T} = \mathbb{R}$ (see [28, p. 49]), the functions $y^{[i]}$, $0 \leqslant i \leqslant 2n$ are the quasi-derivatives of y related to the expression Ly. From (3.2)–(3.4) it follows that

$$y^{[n+j]}(t) = \sum_{i=0}^{j} (-1)^{j-i} (p_i y^{\Delta^{n-i-1}\nabla})^{\nabla^{j-i}}(t), \quad 0 \leqslant j \leqslant n-1, \tag{3.5}$$

$$y^{[2n]}(t) = \sum_{i=0}^{n} (-1)^{n-i} (p_i y^{\Delta^{n-i-1}\nabla})^{\nabla^{n-i-1}\Delta}(t) = Ly(t). \tag{3.6}$$

Definition 3.3. Assume $y, z \in \Omega$ and $t \in \mathbb{T}^*$. The Lagrange bracket of y and z is given by

$$[y, z](t) = \sum_{k=1}^{n} \{y^{[k-1]}(t) z^{[2n-k]}(t) - y^{[2n-k]}(t) z^{[k-1]}(t)\}. \tag{3.7}$$

For $y, z \in \Omega$ and $t \in \mathbb{T}^*$ we also define the bilinear (in y and z) functional F by

$$F[y, z, t] = \sum_{k=1}^{n} y^{[k-1]}(t) z^{[2n-k]}(t), \tag{3.8}$$

so that

$$[y, z](t) = F[y, z, t] - F[z, y, t].$$

Using (3.1) and (3.5) we get that

$$F[y, z, t] = \sum_{k=0}^{n-1} (-1)^k y^{\Delta^{n-k-1}}(t) \sum_{i=0}^{k} (-1)^i (p_i z^{\Delta^{n-i-1}\nabla})^{\nabla^{k-i}}(t). \tag{3.9}$$

Lemma 3.4. *The functional F in* (3.8) *satisfies, for $t \in \mathbb{T}^*$,*

$$F^{\Delta}[y, z, t] = -y(t) Lz(t) + p_n(t) y(t) z(t) + \sum_{k=1}^{n} p_{n-k}^{\sigma}(t) y^{\Delta^k}(t) z^{\Delta^k}(t).$$

Proof. Delta differentiating both sides of (3.8), employing the product rule for delta derivatives, and taking into account the formulas (3.2), (3.4), (3.6), and Theorem 2.11(ii), we get

$$\begin{aligned} F^{\Delta}[y,z,t] &= \sum_{k=1}^{n} y^{[k-1]}(z^{[2n-k]})^{\Delta} + \sum_{k=1}^{n}(y^{[k-1]})^{\Delta}(z^{[2n-k]})^{\sigma} \\ &= y^{[0]}(z^{[2n-1]})^{\Delta} + \sum_{k=2}^{n} y^{[k-1]}(z^{[2n-k]})^{\Delta} + (y^{[n-1]})^{\Delta}(z^{[n]})^{\sigma} + \sum_{k=1}^{n-1}(y^{[k-1]})^{\Delta}(z^{[2n-k]})^{\sigma} \\ &= y(p_n z - Lz) + \sum_{k=2}^{n} y^{[k-1]}(z^{[2n-k]})^{\Delta} + y^{\Delta^n} p_0^{\sigma} z^{\Delta^n} + \sum_{k=2}^{n}(y^{[k-2]})^{\Delta}(z^{[2n-k+1]})^{\sigma}. \end{aligned}$$

Further, by (3.1) we have

$$(y^{[k-2]})^{\Delta} = y^{\Delta^{k-1}} = y^{[k-1]} \quad \text{for } 2 \leqslant k \leqslant n$$

and from (3.3) for z, replacing the k by $n-k+1$, applying the σ operator to both sides and using Theorem 2.11(ii), we find

$$(z^{[2n-k+1]})^{\sigma} = p_{n-k+1}^{\sigma} z^{\Delta^{k-1}} - (z^{[2n-k]})^{\Delta} \quad \text{for } 2 \leqslant k \leqslant n.$$

Consequently we obtain the desired result. □

Theorem 3.5 (*Lagrange Identity*). *If $y, z \in \Omega$, then for $t \in \mathbb{T}^*$,*

$$z(t)Ly(t) - y(t)Lz(t) = [y,z]^{\Delta}(t), \tag{3.10}$$

where $[y,z]$ is the Lagrange bracket of y and z defined by (3.7).

Proof. By (3.7) and (3.8) we have

$$[y,z](t) = F[y,z,t] - F[z,y,t].$$

Delta differentiating both sides and applying Lemma 3.4 we obtain (3.10). □

If we delta integrate both sides of (3.10) from a to b, where $a, b \in \mathbb{T}^*$, then we obtain Lagrange's identity in integral form, also called Green's formula,

$$\int_a^b z(t)Ly(t)\Delta t - \int_a^b y(t)Lz(t)\Delta t = [y,z]_a^b,$$

where $[y,z]_a^b = [y,z](b) - [y,z](a)$. Let $g : \mathbb{T} \to \mathbb{R}$ be a right-dense continuous function. Consider the dynamic equation

$$Ly(t) = g(t) \quad \text{for } t \in \mathbb{T}^*. \tag{3.11}$$

If $y \in \Omega$ and (3.11) holds for y, we say that y is a solution of (3.11). In order to obtain an existence and uniqueness theorem for initial value problems involving (3.11), it is necessary to rewrite (3.11) in the form of an equivalent system of first-order equations.

From (3.1) and (3.4) we have, taking into account (3.6),

$$
\begin{aligned}
&(y^{[k]})^{\Delta} = y^{[k+1]}, \quad 0 \leqslant k \leqslant n-2\\
&(y^{[n-1]})^{\Delta} = y^{\Delta^n},
\end{aligned}
\tag{3.12}
$$

$$(y^{[2n-1]})^{\Delta} = p_n y - Ly. \tag{3.13}$$

Further, using (3.2), Theorem 2.11(ii), and the fact that for any function f differentiable at $t \in \mathbb{T}^{\kappa}$ the equality

$$f(\sigma(t)) = f(t) + \mu_{\sigma}(t) f^{\Delta}(t) \tag{3.14}$$

holds, it follows that

$$y^{\Delta^n}(t) = y^{\Delta^{n-1}\nabla}(\sigma(t)) = \frac{1}{p_0(\sigma(t))} y^{[n]}(\sigma(t)) = \frac{1}{p_0^{\sigma}(t)}(y^{[n]}(t) + \mu_{\sigma}(t)(y^{[n]})^{\Delta}(t)).$$

Therefore (3.12) gives

$$(y^{[n-1]})^{\Delta} = \frac{1}{p_0^{\sigma}}(y^{[n]} + \mu_{\sigma}(y^{[n]})^{\Delta}). \tag{3.15}$$

Next, using (3.3) and again applying Theorem 2.11(ii) and (3.14), we find for $1 \leqslant k \leqslant n-1$

$$
\begin{aligned}
(y^{[n+k-1]})^{\Delta}(t) &= (y^{[n-k-1]})^{\nabla}(\sigma(t)) = p_k(\sigma(t)) y^{\Delta^{n-k-1}\nabla}(\sigma(t)) - y^{[n+k]}(\sigma(t))\\
&= p_k^{\sigma}(t) y^{\Delta^{n-k}}(t) - (y^{[n+k]}(t) + \mu_{\sigma}(t)(y^{[n+k]})^{\Delta}(t)),
\end{aligned}
$$

so that

$$(y^{[n+k-1]})^{\Delta} = p_k^{\sigma} y^{[n-k]} - y^{[n+k]} - \mu_{\sigma}(y^{[n+k]})^{\Delta}, \quad 1 \leqslant k \leqslant n-1. \tag{3.16}$$

Setting $k = n-1$ in (3.16) and using (3.13), we obtain

$$(y^{[2n-2]})^{\Delta} = p_{n-1}^{\sigma} y^{[1]} - y^{[2n-1]} - \mu_{\sigma}(y^{[2n-1]})^{\Delta} = p_{n-1}^{\sigma} - y^{[2n-1]} - \mu_{\sigma}(p_n y^{[0]} - Ly). \tag{3.17}$$

Further, setting $k = n-2$ in (3.16) and using (3.17), we obtain

$$
\begin{aligned}
(y^{[2n-3]})^{\Delta} &= p_{n-2}^{\sigma} y^{[2]} - y^{[2n-2]} - \mu_{\sigma}(y^{[2n-2]})^{\Delta}\\
&= p_{n-2}^{\sigma} y^{[2]} - y^{[2n-2]} - \mu_{\sigma}(p_{n-1}^{\sigma} y^{[1]} - y^{[2n-1]}) + (\mu_{\sigma})^2 (p_n y^{[0]} - Ly).
\end{aligned}
$$

Continuing in this way we find that for $1 \leqslant k \leqslant n-1$,

$$(y^{[n+k-1]})^{\Delta} = (-\mu_{\sigma})^{n-k}(p_n y^{[0]} - Ly) + \sum_{i=0}^{n-k-1} (-\mu_{\sigma})^i (p_{k+i}^{\sigma} y^{[n-k-i]} - y^{[n+k+i]}).$$

In particular, for $k = 1$ we have

$$(y^{[n]})^{\Delta} = (-\mu_{\sigma})^{n-1}(p_n y^{[0]} - Ly) + \sum_{i=0}^{n-2} (-\mu_{\sigma})^i (p_{i+1}^{\sigma} y^{[n-i-1]} - y^{[n+i+1]}).$$

Substituting this into (3.15) we may obtain a desired expression for $(y^{[n-1]})^{\Delta}$. Thus, we have obtained the following system:

$$(y^{[k]})^{\Delta} = y^{[k+1]}, \quad 0 \leqslant k \leqslant n-2,$$

$$(y^{[n-1]})^{\Delta} = \frac{1}{p_0^{\sigma}} y^{[n]} + \frac{\mu_\sigma}{p_0^{\sigma}} \left((-\mu_\sigma)^{n-1}(p_n y^{[0]} - Ly) + \sum_{i=0}^{n-2} (-\mu_\sigma)^i (p_{i+1}^{\sigma} y^{[n-i-1]} - y^{[n+i+1]}) \right),$$

$$(y^{[n+k-1]})^{\Delta} = (-\mu_\sigma)^{n-k}(p_n y^{[0]} - Ly) + \sum_{i=0}^{n-k-1} (-\mu_\sigma)^i (p_{k+i}^{\sigma} y^{[n-k-i]} - y^{[n+k+i]}),$$
$$1 \leqslant k \leqslant n-1,$$

$$(y^{[2n-1]})^{\Delta} = p_n y^{[0]} - Ly.$$

Let

$$\vec{y} = [y^{[0]}, y^{[1]}, \ldots, y^{[2n-1]}]^{\mathrm{T}}$$
$$\vec{g} = \left[0, \ldots, 0, \frac{(-\mu_\sigma)^n}{p_0^{\sigma}} g, \mu_\sigma g, -(\mu_\sigma)^2 g, \ldots, -(-\mu_\sigma)^{n-1} g, -g\right]^{\mathrm{T}},$$

where T indicates transpose. In addition, define the matrix functions

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ \frac{-(-\mu_\sigma)^n p_n}{p_0^{\sigma}} & \frac{-(-\mu_\sigma)^{n-1} p_{n-1}^{\sigma}}{p_0^{\sigma}} & \frac{-(-\mu_\sigma)^{n-2} p_{n-2}^{\sigma}}{p_0^{\sigma}} & \frac{-(-\mu_\sigma)^{n-3} p_{n-3}^{\sigma}}{p_0^{\sigma}} & \cdots & \frac{-\mu_\sigma^2 p_2^{\sigma}}{p_0^{\sigma}} & \frac{\mu_\sigma p_1^{\sigma}}{p_0^{\sigma}} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{p_0^{\sigma}} & \frac{-\mu_\sigma}{p_0^{\sigma}} & \frac{(-\mu_\sigma)^2}{p_0^{\sigma}} & \cdots & \frac{(-\mu_\sigma)^{n-3}}{p_0^{\sigma}} & \frac{(-\mu_\sigma)^{n-2}}{p_0^{\sigma}} & \frac{(-\mu_\sigma)^{n-1}}{p_0^{\sigma}} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} (-\mu_\sigma)^{n-1} p_n & (-\mu_\sigma)^{n-2} p_{n-1}^{\sigma} & (-\mu_\sigma)^{n-3} p_{n-2}^{\sigma} & (-\mu_\sigma)^{n-4} p_{n-3}^{\sigma} & \cdots & \mu_\sigma p_2^{\sigma} & p_1^{\sigma} \\ (-\mu_\sigma)^{n-2} p_n & (-\mu_\sigma)^{n-3} p_{n-1}^{\sigma} & (-\mu_\sigma)^{n-4} p_{n-2}^{\sigma} & (-\mu_\sigma)^{n-5} p_{n-3}^{\sigma} & \cdots & p_2^{\sigma} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (-\mu_\sigma)^2 p_n & -\mu_\sigma p_{n-1}^{\sigma} & p_{n-2}^{\sigma} & 0 & \cdots & 0 & 0 \\ -\mu_\sigma p_n & p_{n-1}^{\sigma} & 0 & 0 & \cdots & 0 & 0 \\ p_n & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} -1 & \mu_\sigma & \cdots & -(-\mu_\sigma)^{n-4} & -(-\mu_\sigma)^{n-3} & -(-\mu_\sigma)^{n-2} \\ 0 & -1 & \cdots & -(-\mu_\sigma)^{n-5} & -(-\mu_\sigma)^{n-4} & -(-\mu_\sigma)^{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \mu_\sigma & -(-\mu_\sigma)^2 \\ 0 & 0 & \cdots & 0 & -1 & \mu_\sigma \\ 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

so that

$$A(t) = \begin{bmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{bmatrix}$$

is an $(2n) \times (2n)$ variable matrix function on $\mathbb{T}^*$. Thus, we obtain that the dynamic equation (3.11) is equivalent to the first-order system

$$\vec{y}^{\Delta} = A(t)\vec{y} + \vec{g}(t) \quad \text{for } t \in \mathbb{T}^*. \tag{3.18}$$

We are now able to prove the following theorem.

Theorem 3.6 (*Existence and Uniqueness*). *Let $t_0 \in \mathbb{T}^*$ be fixed and c_i, $0 \leqslant i \leqslant 2n-1$, be given real constants. Then Eq.* (3.11) *has a unique solution $y : \mathbb{T} \to \mathbb{R}$ such that*

$$y^{[k]}(t_0) = c_k, \quad 0 \leqslant k \leqslant 2n-1.$$

Proof. Since Eq. (3.11) is equivalent to the system (3.18), it is sufficient [13, Theorem 5.8] to show that

$$I + \mu_\sigma(t)A(t)$$

is invertible for all $t \in \mathbb{T}^*$. To calculate $\det(I + \mu_\sigma A)$, we multiply the $2n$th column of $I + \mu_\sigma A$ by $-\mu_\sigma p_n$ and add it to the first column, multiply the $(2n-i)$th column by $-\mu_\sigma p^\sigma_{n-i}$ and add it to the $(i+1)$st column for all $1 \leqslant i \leqslant n-1$. Then we get a lower triangular matrix whose main diagonal elements all have the value 1. Therefore, $\det(I + \mu_\sigma A) = 1 \neq 0$, concluding the proof. □

Consider the homogeneous equation $Ly(t) = 0$.

Definition 3.7. Let y_i, $1 \leqslant i \leqslant 2n$, be solutions of $Ly(t) = 0$. The Wronskian of these solutions is defined to be the determinant

$$W_t(y_1, \ldots, y_{2n}) = \begin{vmatrix} y_1 & y_2 & \cdots & y_{2n} \\ y_1^{[1]} & y_2^{[1]} & \cdots & y_{2n}^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{[2n-1]} & y_2^{[2n-1]} & \cdots & y_{2n}^{[2n-1]} \end{vmatrix}.$$

The proofs of the following two theorems follow in the same manner as the differential equations case; see [28, pp. 57–58].

Theorem 3.8. *If the solutions y_i, $1 \leqslant i \leqslant 2n$, of the homogeneous equation $Ly = 0$ are linearly dependent, then their Wronskian vanishes identically on $\mathbb{T}^*$. Conversely, if the Wronskian vanishes at at least one point in $\mathbb{T}^*$, then the solutions y_i, $1 \leqslant i \leqslant 2n$, are linearly dependent.*

We can easily construct a linearly independent system of solutions y_i, $1 \leqslant i \leqslant 2n$, of a homogeneous system. We need only choose a system of solutions which satisfy initial conditions of the form

$$y_j^{[k-1]}(t_0) = a_{jk}, \quad 1 \leqslant j, \quad k \leqslant 2n,$$

where the determinant of the matrix $[a_{jk}]$ is different from zero. A linearly independent system of solutions y_i, $1 \leqslant i \leqslant 2n$, is called a fundamental system.

Theorem 3.9. *Every solution of a homogeneous equation is a linear combination of a fixed, arbitrarily chosen, fundamental system.*

4. Self-adjoint boundary conditions and Green's functions

Let $a, b \in \mathbb{T}$ be such that $a \in \mathbb{T}^{\kappa^n}$, $b \in \mathbb{T}_{\kappa^n}$, and $a < b$. If y and z are real-valued right-dense continuous functions and bounded on $[a, b)$, define their inner product to be

$$\langle y, z \rangle = \int_a^b y(t)z(t)\Delta t.$$

Suppose that $p_n : [a, b) \to \mathbb{R}$ is a right-dense continuous and bounded function, and for $0 \leqslant i \leqslant n-1$, $p_i : [\rho^{n-i-1}(a), b] \to \mathbb{R}$ is right-dense continuous with $p_0(t) \neq 0$ on $[\rho^{n-1}(a), b]$.

Definition 4.1. Denote by $\Omega[a, b)$ the linear set of all right-dense continuous functions y: $[\rho^n(a), \sigma^{n-1}(b)] \to \mathbb{R}$, such that

(i) for $0 \leqslant i \leqslant n-1$ the function $(p_i y^{\Delta^{n-i-1}\nabla})^{\nabla^{n-i-1}}(t)$ is defined for $t \in [a, b]$,
(ii) for $0 \leqslant i \leqslant n-1$ the function $(p_i y^{\Delta^{n-i-1}\nabla})^{\nabla^{n-i-1}\Delta}(t)$ is defined for $t \in [a, b)$ and is right-dense continuous and bounded on $[a, b)$.

For $y \in \Omega[a, b)$ let

$$Ly(t) = \sum_{i=0}^{n} (-1)^{n-i} (p_i y^{\Delta^{n-i-1}\nabla})^{\nabla^{n-i-1}\Delta}(t), \quad t \in [a, b). \tag{4.1}$$

Then Ly is right-dense continuous and bounded on $[a, b)$. Together with the dynamic equation (4.1) define the boundary conditions

$$U_j(y) := \sum_{k=1}^{2n} \alpha_{jk} y^{[k-1]}(a) + \sum_{k=1}^{2n} \beta_{jk} y^{[k-1]}(b) = 0, \quad 1 \leqslant j \leqslant 2n, \tag{4.2}$$

where α_{jk}, β_{jk}, $1 \leqslant k, j \leqslant 2n$ are given real numbers.

Definition 4.2. The boundary conditions (4.2) are self-adjoint with respect to the dynamic equation (4.1), if and only if

$$\langle Ly, z\rangle = \langle y, Lz\rangle \tag{4.3}$$

for all functions $y, z \in \Omega[a, b)$ satisfying the boundary conditions (4.2).

By the Lagrange identity (3.10) we have, for all $y, z \in \Omega[a, b)$,

$$\langle Ly, z\rangle - \langle y, Lz\rangle = [y, z]_a^b,$$

where $[y, z]$ is as defined previously. Therefore, boundary conditions (4.2) are self-adjoint , if and only if

$$[y, z]_a^b = 0$$

for all functions $y, z \in \Omega[a, b)$ satisfying (4.2). For example the boundary conditions

$$y^{[k]}(a) = y^{[k]}(b) = 0, \quad 0 \leqslant k \leqslant n-1$$

and also the boundary conditions

$$y^{[k]}(a) = y^{[k]}(b), \quad 0 \leqslant k \leqslant 2n-1,$$

are self-adjoint. The BVP $Ly(t) = 0$, $U_j(y) = 0$, $1 \leqslant j \leqslant 2n$ has a Green's function $G(t, s)$ if for any right-dense continuous and bounded function $g : [a, b) \to \mathbb{R}$ the non-homogeneous BVP $Ly(t) = g(t)$, $U_j(y) = 0$, $1 \leqslant j \leqslant 2n$, has a unique solution $y : [\rho^n(a), \sigma^{n-1}(b)] \to \mathbb{R}$ which is given by

$$y(t) = \int_a^b G(t, s)g(s)\Delta s.$$

Let Λ be a dynamic operator generated by the dynamic expression Ly and the boundary conditions $U_j(y) = 0$, $1 \leqslant j \leqslant 2n$. Then the domain of definition $D(\Lambda)$ of the operator Λ consists of all functions $y \in \Omega[a, b)$ satisfying the boundary conditions (4.2), and $\Lambda y = Ly$ for all $y \in D(\Lambda)$. Existence of the Green function $G(t, s)$ for $Ly(t) = 0$, $U_j(y) = 0$, $1 \leqslant j \leqslant 2n$, means that the corresponding operator Λ has an inverse Λ^{-1} given by

$$(\Lambda^{-1}g)(t) = \int_a^b G(t, s)g(s)\Delta s, \quad t \in [\rho^n(a), \sigma^{n-1}(b)], \tag{4.4}$$

for all bounded, right-dense continuous functions $g : [a, b) \to \mathbb{R}$.

Suppose that the boundary conditions (4.2) are self-adjoint with respect to Ly. Then (4.3) implies that the operator Λ is self-adjoint (symmetric):

$$\langle \Lambda y, z\rangle = \langle y, \Lambda z\rangle \quad \text{for all } y, z \in D(\Lambda).$$

It easily follows that the inverse operator Λ^{-1} (provided it exists) is also symmetric:

$$\langle \Lambda^{-1} f, g\rangle = \langle f, \Lambda^{-1} g\rangle \quad \text{for all bounded } f, g \in C_{\mathrm{rd}}[a, b). \tag{4.5}$$

Now (4.4) and (4.5) yield that the Green function $G(t, s)$, provided it exists, of the self-adjoint BVP $Ly(t) = 0$, $U_j(y) = 0$, $1 \leqslant j \leqslant 2n$, must be symmetric, i.e.

$$G(t, s) = G(s, t) \quad \text{for } t, s \in [a, b).$$

5. Self-adjoint equations as Hamiltonian systems

Let us show that the $2n$th-order self-adjoint dynamic equation $Ly=0$ in which Ly is of the form (1.1) can be written as an equivalent Hamiltonian system.

For any function $y \in \Omega$ we have from (3.1) and (3.4),

$$\begin{aligned}
&(y^{[k]})^{\Delta} = y^{[k+1]}, \quad 0 \leqslant k \leqslant n-2,\\
&(y^{[n-1]})^{\Delta} = y^{\Delta^n},\\
&(y^{[2n-1]})^{\Delta} = p_n y^{[0]} - Ly.
\end{aligned}$$

Further applying Theorem 2.11(ii) and using (3.2) we get

$$y^{\Delta^n}(t) = y^{\Delta^{n-1}\nabla}(\sigma(t)) = \frac{1}{p_0(\sigma(t))} y^{[n]}(\sigma(t)).$$

Applying again Theorem 2.11(ii) and (3.3) we find for $1 \leqslant k \leqslant n-1$,

$$\begin{aligned}
(y^{[n+k-1]})^{\Delta}(t) &= (y^{[n+k-1]})^{\nabla}(\sigma(t)) = p_k(\sigma(t)) y^{\Delta^{n-k-1}\nabla}(\sigma(t)) - y^{[n+k]}(\sigma(t))\\
&= p_k(\sigma(t)) y^{\Delta^{n-k}}(t) - y^{[n+k]}(\sigma(t)) = p_k(\sigma(t)) y^{[n-k]}(t) - y^{[n+k]}(\sigma(t)).
\end{aligned}$$

Thus we have obtained the following system of relations for any function $y \in \Omega$:

$$\begin{aligned}
&(y^{[k]})^{\Delta} = y^{[k+1]}, \quad 0 \leqslant k \leqslant n-2,\\
&(y^{[n-1]})^{\Delta} = \frac{1}{p_0^{\sigma}} (y^{[n]})^{\sigma},\\
&(y^{[n-k-1]})^{\Delta} = p_k^{\sigma} y^{[n-k]} - (y^{[n+k]})^{\sigma}, \quad 1 \leqslant k \leqslant n-1,\\
&(y^{[2n-1]})^{\Delta} = p_n y^{[0]} - Ly.
\end{aligned}$$

Therefore setting

$$\vec{y}(t) = \begin{pmatrix} y^{[2n-1]}(t) \\ y^{[2n-2]}(t) \\ \vdots \\ y^{[n]}(t) \end{pmatrix}, \quad \vec{u}(t) = \begin{pmatrix} y^{[0]}(t) \\ y^{[1]}(t) \\ \vdots \\ y^{[n-1]}(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix},$$

$$B(t) = \begin{pmatrix} p_n(t) & 0 & 0 & \cdots & 0 & 0 \\ 0 & p_{n-1}^{\sigma}(t) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_2^{\sigma}(t) & 0 \\ 0 & 0 & 0 & \cdots & 0 & p_1^{\sigma}(t) \end{pmatrix}, \quad C(t) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{p_0^{\sigma}(t)} \end{pmatrix},$$

we get that the equation $Ly(t)=0$, $t \in \mathbb{T}^*$ is equivalent to the linear Hamiltonian system on time scales [1,22],

$$\vec{y}^{\Delta}(t) = A(t)\vec{y}^{\sigma}(t) + B(t)\vec{u}(t), \quad \vec{u}^{\Delta}(t) = C(t)\vec{y}^{\sigma}(t) - A^{\mathrm{T}}(t)\vec{u}(t). \tag{5.1}$$

Note that $I - \mu_{\sigma}(t)A(t)$ is invertible. For a discrete analogue see [11].

Now, let us present some properties of solutions to the homogeneous equation $Ly(t)=0$, $t\in\mathbb{T}^*$. From the Lagrange identity (3.10) we immediately get the following theorem.

Theorem 5.1. *If y and z are solutions of $Ly(t)=0$, $t\in\mathbb{T}^*$, then the Lagrange bracket of y and z satisfies*

$$[y,z](t)\equiv \textit{constant}.$$

Lemma 3.4 yields the following result.

Theorem 5.2. *Let $F[y,z,t]$ be defined as in* (3.8) (*see also* (3.9)). *If y is a solution of $Ly(t)=0$, $t\in\mathbb{T}^*$, then*

$$F^{\Delta}[y,y,t]=p_n(t)(y(t))^2+\sum_{k=1}^{n}p_{n-k}^{\sigma}(t)(y^{\Delta^k}(t))^2,\quad t\in\mathbb{T}^*.$$

In particular, if $p_n(t)\geqslant 0$, $p_i^{\sigma}(t)\geqslant 0$ for $0\leqslant i\leqslant n-1$ and $t\in\mathbb{T}^*$, then $F[y,y,t]$ is non-decreasing along solutions of $Ly(t)=0$, $t\in\mathbb{T}^*$.

Lemma 5.3. *Assume $\eta\in\Omega[a,b)$. Then*

$$F[\eta,\eta,b]-F[\eta,\eta,a]=-\langle\eta,L\eta\rangle+\int_a^b\left(p_n(t)(\eta(t))^2+\sum_{k=1}^{n}p_{n-k}^{\sigma}(t)\left(\eta^{\Delta^k}(t)\right)^2\right)\Delta t.\tag{5.2}$$

Proof. Setting $y=z=\eta$ in Lemma 3.4 and then delta integrating both sides from a to b we get the desired result. □

Definition 5.4. The set of admissible variations is given by

$$\mathscr{S}=\{\eta\in\Omega[a,b):\eta^{\Delta^k}(a)=\eta^{\Delta^k}(b)=0,\ \ 0\leqslant k\leqslant n-1\},$$

with corresponding functional

$$\mathscr{F}(\eta)=\int_a^b\left(p_n(t)(\eta(t))^2+\sum_{k=1}^{n}p_{n-k}^{\sigma}(t)(\eta^{\Delta^k}(t))^2\right)\Delta t.\tag{5.3}$$

For an admissible variation $\eta\in\mathscr{S}$, Lemma 5.3 implies that

$$\mathscr{F}(\eta)=\langle\eta,L\eta\rangle.$$

The functional $\mathscr{F}$ is positive definite on the set of admissible variations $\mathscr{S}$ if $\mathscr{F}(\eta)\geqslant 0$ for all $\eta\in\mathscr{S}$, and $\mathscr{F}(\eta)=0$ if and only if $\eta=0$.

Note that the bilinear functional F in (3.8) and the vector valued functions $\vec{y}$ and $\vec{u}$ given above satisfy the dot-product equation

$$(\vec{y}\cdot\vec{u})(t)=F[y,y,t].$$

Theorem 5.5. *Assume $p_n(t) \geqslant 0$, $p_i^\sigma(t) \geqslant 0$ for $0 \leqslant i \leqslant n-1$ and $t \in \mathbb{T}^*$, and $p_0^\sigma(t) > 0$ for $t \in \mathbb{T}^*$. Then the functional $\mathscr{F}$ is positive definite on $\mathscr{S}$, and the linear Hamiltonian system* (5.1) *being considered for $t \in [a, b)$ is disconjugate on $[a, b]$. In particular the self-adjoint BVP*

$$Ly(t) = 0, \quad t \in [a, b),$$
$$y^{\Delta^j}(a) = y^{\Delta^j}(b) = 0, \quad j = 0, 1, \ldots, n-1$$

has only the trivial solution.

Proof. Let $t \in \mathbb{T}^*$. From $p_n(t) \geqslant 0$, $p_i^\sigma(t) \geqslant 0$ for $0 \leqslant i \leqslant n-1$, and (5.3), it is clear that $\mathscr{F}(\eta) \geqslant 0$ for all $\eta \in \mathscr{S}$, and that $\mathscr{F}(\eta) = 0$ if $\eta = 0$. Now suppose $\eta \in \mathscr{S}$ and $F(\eta) = 0$. Then

$$0 = \int_a^b \left(p_n(t)(\eta(t))^2 + \sum_{k=1}^n p_{n-k}^\sigma(t)(\eta^{\Delta^k}(t))^2 \right) \Delta t \geqslant \int_a^b p_0^\sigma(t)(\eta^{\Delta^n}(t))^2 \Delta t$$

and since $p_0^\sigma(t) > 0$, we have that $\eta^{\Delta^n}(t) = 0$ for $t \in [a, b)$. Because η is admissible, it solves the initial-value problem

$$\eta^{\Delta^n}(t) = 0, \quad t \in [a, b),$$
$$\eta^{\Delta^k}(a) = 0, \quad 0 \leqslant k \leqslant n-1.$$

By uniqueness of solutions to initial value problems, η is the trivial solution in the set of admissible functions, whence $\mathscr{F}$ is positive definite on that set. By (5.3), if y is a solution of $Ly(t) = 0$, $t \in [a, b)$, then

$$\begin{aligned} F[y, y, b] - F[y, y, a] &= (\vec{y} \cdot \vec{u})(b) - (\vec{y} \cdot \vec{u})(a) \\ &= \int_a^b \left(p_n(t)(y(t))^2 + \sum_{k=1}^n p_{n-k}^\sigma(t)(y^{\Delta^k}(t))^2 \right) \Delta t = \mathscr{F}(y). \end{aligned}$$

Note that the Hamiltonian system (5.1) is disconjugate on $[a, b]$ if and only if for a vector solution $\vec{y}$, $\vec{u}$ of (5.1), the following is positive definite:

$$\int_a^b (\vec{y}^{\mathrm{T}}(\sigma(t))C(t)\vec{y}(\sigma(t)) + \vec{u}^{\mathrm{T}}(t)B(t)\vec{u}(t))\Delta t = \mathscr{F}(y). \qquad \square$$

Following [12,16], the point $t = a$ is a generalized zero of order (at least) n of y if

$$y^{\Delta^j}(a) = 0, \quad j = 0, 1, \ldots, n-1.$$

The point $t \in \mathbb{T}^{\kappa^{n-1}}$, $t > a$, is a generalized zero of order (at least) n of y if $y^{\Delta^j}(t) = 0$, $j = 0, 1, \ldots, n-1$, or

$$y^{\Delta^j}(t) = 0, \quad j = 0, 1, \ldots, n-2, \qquad y^{\Delta^{n-1}}(\rho(t))y^{\Delta^{n-1}}(t) < 0.$$

In the second case t is left-scattered. The equation $Ly = 0$ is (n, n) disconjugate on $[a, b]$ provided there is no non-trivial solution of $Ly = 0$ with a zero of order (at least) n in $(a, b]$ preceded by a generalized zero of order (at least) n in $[a, b)$. These ideas lead to the next conclusion.

Theorem 5.6. *If $p_0(t) > 0$ on $[a, b]$, then $Ly(t) = 0$ is (n, n) disconjugate on $[a, b]$.*

Proof. Suppose y is a solution of $Ly = 0$, and without loss of generality assume y has a zero of order n at b, namely $y^{\Delta^j}(b) = 0$, $j = 0, 1, \ldots, n-1$. Then from (3.9) we have $F[y, y, b] = 0$, and $F[y, y, t] \leqslant 0$ for all $t \in [a, b)$ by Theorem 5.2. If y has a generalized zero at $z \in [a, b)$ of order n such that

$$y^{\Delta^j}(z) = 0, \quad j = 0, 1, \ldots, n-2, \qquad y^{\Delta^{n-1}}(\rho(z))y^{\Delta^{n-1}}(z) < 0,$$

then z is left-scattered. This, however, means that

$$\begin{aligned} F[y, y, z] &= \sum_{k=0}^{n-1} (-1)^k y^{\Delta^{n-k-1}}(z) \sum_{i=0}^{k} (-1)^i \, (p_i y^{\Delta^{n-i-1}\nabla})^{\nabla^{k-i}}(z) \\ &= y^{\Delta^{n-1}}(z) p_0(z) \left(\frac{y^{\Delta^{n-1}}(z) - y^{\Delta^{n-1}}(\rho(z))}{z - \rho(z)} \right) \\ &= \frac{p_0(z)}{z - \rho(z)} [(y^{\Delta^{n-1}}(z))^2 - y^{\Delta^{n-1}}(\rho(z)) y^{\Delta^{n-1}}(z)] > 0, \end{aligned}$$

a contradiction. Therefore z must be a generalized zero of the first kind, namely

$$y^{\Delta^j}(z) = 0, \quad j = 0, 1, \ldots, n-1.$$

But then y is a trivial solution of $Ly = 0$ by the previous theorem. □

6. Second-order dynamic equations

Taking $n = 1$, we find

$$Ly(t) = -(p_0 y^{\nabla})^{\Delta}(t) + p_1(t)y(t)$$

for $t \in \mathbb{T}^* = \mathbb{T}_\kappa^\kappa$, and for each function $y \in \Omega$,

$$y^{[0]} = y, \quad y^{[1]} = p_0 y^{\nabla}, \quad y^{[2]} = p_1 y - (y^{[1]})^{\Delta}.$$

Then

$$Ly = y^{[2]}$$

as expected. In addition, the dynamic equation $Ly(t) = g(t)$ for $t \in \mathbb{T}^*$ is equivalent to the first-order system

$$\vec{y}^{\Delta} = A(t)\vec{y} + \vec{g}(t), \quad t \in \mathbb{T}^*,$$

where

$$\vec{y} = \begin{bmatrix} y^{[0]} \\ y^{[1]} \end{bmatrix}, \quad \vec{g} = \begin{bmatrix} -\frac{\mu_\sigma}{p_0^\sigma} g \\ -g \end{bmatrix}, \quad A(t) = \begin{bmatrix} \mu_\sigma(t) \frac{p_1(t)}{p_0^\sigma(t)} & \frac{1}{p_0^\sigma(t)} \\ p_1(t) & 0 \end{bmatrix}.$$

It can easily be seen that $\det(I + \mu_\sigma(t)A(t)) = 1 \neq 0$. The Wronskian of two solutions $y_1(t), y_2(t)$, is given by

$$W_t(y_1, y_2) = \begin{vmatrix} y_1^{[0]}(t) & y_2^{[0]}(t) \\ y_1^{[1]}(t) & y_2^{[1]}(t) \end{vmatrix} = p_0(t)(y_1(t)y_2^\nabla(t) - y_2^\nabla(t)y_2(t)),$$

which coincides with the Lagrange bracket, giving rise to the following theorem.

Theorem 6.1. *The Wronskian of any two solutions of $Ly(t) = 0$ is independent of t.*

The following theorem presents a variation of constants formula for the non-homogeneous equation $Ly(t) = g(t)$.

Theorem 6.2. *Suppose that y_1, y_2 form a fundamental system of solutions of the homogeneous equation $Ly(t) = 0$ and $w = W_t(y_1, y_2)$. Then the general solution of the non-homogeneous equation $Ly(t) = g(t)$ is given by*

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \frac{1}{w}\int_{t_0}^{t} (y_1(t)y_2(s) - y_1(s)y_2(t))g(s)\Delta s,$$

where $t_0 \in \mathbb{T}^*$ and c_1, c_2 are real constants.

Proof. It suffices to show that the function

$$z(t) = \frac{1}{w}\int_{t_0}^{t} (y_1(t)y_2(s) - y_1(s)y_2(t))g(s)\Delta s$$

is a particular solution of the non-homogeneous equation $Ly(t) = g(t)$. Nabla differentiating both sides yields

$$z^\nabla(t) = \frac{1}{w}\int_{t_0}^{t} (y_1^\nabla(t)y_2(s) - y_1(s)y_2^\nabla(t))g(s)\Delta s.$$

Hence

$$\begin{aligned}(p_0 z^\nabla)^\Delta(t) &= \frac{1}{w}(p_0^\sigma(t)y_1^{\nabla\sigma}(t)y_2(t) - y_1(t)p_0^\sigma(t)y_2^{\nabla\sigma}(t))g(t) \\ &\quad + \frac{1}{w}\int_{t_0}^{t} ((p_0 y_1^\nabla)^\Delta(t)y_2(s) - y_1(s)(p_0 y_2^\nabla)^\Delta(t))g(s)\Delta s. \qquad (6.1)\end{aligned}$$

On the other hand,

$$p_0^\sigma(t)y_1^\nabla(\sigma(t))y_2(t) - y_1(t)p_0^\sigma(t)y_2^\nabla(\sigma(t)) = -W_t(y_1, y_2) = -w.$$

If $\sigma(t) = t$ then the result is clear. If $\sigma(t) > t$, then we use the formulas

$$y_i(t) = y_i(\sigma(t)) - \mu_\sigma(t)y_i^\nabla(\sigma(t)), \quad i = 1, 2,$$

to find

$$p_0^\sigma(t)y_1^\nabla(\sigma(t))y_2(t) - y_1(t)p_0^\sigma(t)y_2^\nabla(\sigma(t)) = p_0^\sigma(t)(y_1^\nabla(\sigma(t))y_2(\sigma(t)) - y_1(\sigma(t))y_2^\nabla(\sigma(t))) \\ = -W_{\sigma(t)}(y_1, y_2) = -w.$$

Therefore from (6.1) and the fact that

$$(p_0 y_i^\nabla)^\Delta(t) = p_1(t)y_i(t), \quad i = 1, 2,$$

we obtain

$$(p_0 z^\nabla)^\Delta(t) = -g(t) + p_2(t)z(t),$$

that is $z(t)$ satisfies $Ly(t) = g(t)$. □

For $y \in \Omega[a, b)$ let

$$Ly(t) = -(p_0 y^\nabla)^\Delta(t) + p_1(t)y(t), \quad t \in [a, b)$$

together with the boundary conditions

$$\begin{aligned} \alpha_{11}y(a) + \alpha_{12}y^{[1]}(a) + \beta_{11}y(b) + \beta_{12}y^{[1]}(b) = 0, \\ \alpha_{21}y(a) + \alpha_{22}y^{[1]}(a) + \beta_{21}y(b) + \beta_{22}y^{[1]}(b) = 0, \end{aligned} \tag{6.2}$$

where α_{jk}, β_{jk} are given real numbers, $j, k = 1, 2$. Set

$$N = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12} \\ \alpha_{21} & \alpha_{22} & \beta_{21} & \beta_{22} \end{bmatrix}.$$

We will assume that the matrix N has rank 2. This means that the two boundary conditions (6.2) are linearly independent. As before, we call the boundary conditions (6.2) self-adjoint with respect to the dynamic expression Ly if

$$\langle Ly, z\rangle - \langle y, Lz\rangle = [y, z]_a^b$$

for all functions $y, z \in \Omega[a, b)$ satisfying the boundary conditions (6.2). Recall that by Green's formula, the boundary conditions (6.2) are self-adjoint if and only if

$$[y, z]_a^b = 0.$$

Set

$$N_1 = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}, \quad N_2 = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}.$$

Theorem 6.3. *If* $\det N_1 = \det N_2$ *then the boundary conditions* (6.2) *are self-adjoint*.

Proof. Let $y, z \in \Omega[a, b)$, be functions which satisfy boundary conditions (6.2). Then we have

$$N_1 \begin{bmatrix} y(a) & z(a) \\ y^{[1]}(a) & z^{[1]}(a) \end{bmatrix} = N_2 \begin{bmatrix} -y(b) & -z(b) \\ -y^{[1]}(b) & -z^{[1]}(b) \end{bmatrix}.$$

Passing to determinants we have

$$(\det N_1)[y, z](a) = (\det N_2)[y, z](b).$$

Suppose that $\det N_1 \neq 0$. Then $\det N_2 \neq 0$, which implies $[y, z](a) = [y, z](b)$, i.e., $[y, z]_a^b = 0$. Suppose that $\det N_1 = 0$. Then $\det N_2 = 0$ as well. Since N has rank 2, it is clear that the boundary conditions (6.2) are equivalent to separated boundary conditions of the form

$$\begin{aligned} &\alpha_1 y(a) + \alpha_2 y^{[1]}(a) = 0, \quad |\alpha_1| + |\alpha_2| \neq 0, \\ &\beta_1 y(b) + \beta_2 y^{[1]}(b) = 0, \quad |\beta_1| + |\beta_2| \neq 0, \end{aligned} \tag{6.3}$$

where α_i, β_i, $i = 1, 2$ are real numbers. It can easily be verified that for any functions $y, z \in \Omega[a, b)$ satisfying boundary conditions (6.2) we have

$$[y, z](a) = [y, z](b) = 0,$$

completing the proof. $\square$

Remark 6.4. As was noted above, the separated boundary conditions (6.3), in particular, the boundary conditions $y(a) = y(b) = 0$ are self-adjoint. The "periodic" boundary conditions $y(a) = y(b)$, $y^{[1]}(a) = y^{[1]}(b)$ which are non-separated, are also self-adjoint.

We will now construct the Green function for the self-adjoint BVP

$$-(p_0 y^{\nabla})^{\Delta}(t) + p_1(t)y(t) = g(t) \tag{6.4}$$

$$\alpha y(a) - \beta y^{[1]}(a) = 0, \quad \gamma y(b) + \delta y^{[1]}(b) = 0, \tag{6.5}$$

where $\alpha, \beta, \gamma, \delta$ are real numbers, such that $|\alpha| + |\beta| \neq 0$, $|\gamma| + |\delta| \neq 0$.

Remark 6.5. The minus sign on the left-hand side of (6.4), as well as in the first boundary condition of (6.5), is taken so that the positivity of the Green function can be formulated in terms of $p_0(t) > 0$, $p_1(t) \geqslant 0$, $\alpha, \beta, \gamma, \delta \geqslant 0$ (see [9]).

Denote by ψ and ϕ the solutions of the corresponding homogeneous equation

$$-(p_0 y^{\nabla})^{\Delta}(t) + p_1(t)y(t) = 0, \quad t \in [a, b), \tag{6.6}$$

under the initial conditions

$$\phi(a) = \beta, \quad \phi^{[1]}(a) = \alpha, \tag{6.7}$$

$$\psi(b) = \delta, \quad \psi^{[1]}(b) = -\gamma, \tag{6.8}$$

so that ϕ and ψ satisfy the first and second boundary conditions in (6.5), respectively. Set

$$w = W_t(\phi, \psi) = \phi(t)\psi^{[1]}(t) - \phi^{[1]}(t)\psi(t).$$

Since the Wronskian of any two solutions is independent of t, evaluating at $t = a$, $t = b$, and using the boundary conditions (6.7), (6.8) yields

$$w = \beta\psi^{[1]}(a) - \alpha\psi(a) = -\gamma\phi(b) - \delta\phi^{[1]}(b).$$

In addition $w \neq 0$, if and only if the homogeneous equation (6.6) has only the trivial solution satisfying the boundary conditions (6.5).

Theorem 6.6. *If* $w \neq 0$, *then the non-homogeneous BVP* (6.4), (6.5), *has a unique solution y for which the formula*

$$y(t) = \int_a^b G(t,s)g(s)\Delta s, \quad t \in [\rho(a), b]$$

holds, where the function $G(t,s)$ *is given by*

$$G(t,s) = -\frac{1}{w}\begin{cases} \phi(t)\psi(s), & \rho(a) \leqslant t \leqslant s \leqslant b, \\ \phi(s)\psi(t), & \rho(a) \leqslant s \leqslant t \leqslant b \end{cases}$$

and $G(t,s)$ *is the Green function of the BVP* (6.4), (6.5). *Furthermore the Green function is symmetric, that is* $G(t,s) = G(s,t)$ *for* $t, s \in [\rho(a), b]$.

Proof. Since $w \neq 0$, the solutions ϕ and ψ of the homogeneous equation (6.6) are linearly independent. Thus the general solution of the non-homogeneous equation (6.4) has the form

$$y(t) = c_1\phi(t) + c_2\psi(t) + \frac{1}{w}\int_a^t (\phi(t)\psi(s) - \phi(s)\psi(t))g(s)\Delta s, \tag{6.9}$$

where c_1 and c_2 are real constants. We now construct c_1 and c_2 so that the function y satisfies the boundary conditions (6.2). Using (6.9) we have

$$y^{[1]}(t) = c_1\phi^{[1]}(t) + c_2\psi^{[1]}(t) + \frac{1}{w}\int_a^t (\phi^{[1]}(t)\psi(s) - \phi(s)\psi^{[1]}(t))g(s)\Delta s. \tag{6.10}$$

Consequently,

$$y(a) = c_1\phi(a) + c_2\psi(a) = c_1\beta + c_2\psi(a),$$

$$y^{[1]}(a) = c_1\phi^{[1]}(a) + c_2\psi^{[1]}(a) = c_1\alpha + c_2\psi^{[1]}(a).$$

Substituting these values of $y(a)$ and $y^{[1]}(a)$ into the first condition of (6.5) we have

$$c_2(\alpha\psi(a) - \beta\psi^{[1]}(a)) = 0.$$

On the other hand, using the definition of w,

$$\alpha\psi(a) - \beta\psi^{[1]}(a) = -w \neq 0.$$

Consequently $c_2 = 0$, and (6.9), (6.10), take the form

$$y(t) = c_1\phi(t) + \frac{1}{w}\int_a^t (\phi(t)\psi(s) - \phi(s)\psi(t))g(s)\Delta s,$$

$$y^{[1]}(t) = c_1\phi^{[1]}(t) + \frac{1}{w}\int_a^t (\phi^{[1]}(t)\psi(s) - \phi(s)\psi^{[1]}(t))g(s)\Delta s.$$

Hence

$$y(b) = c_1\phi(b) + \frac{1}{w}\int_a^b (\phi(b)\psi(s) - \phi(s)\psi(b))g(s)\Delta s,$$

$$y^{[1]}(b) = c_1\phi^{[1]}(b) + \frac{1}{w}\int_a^b (\phi^{[1]}(b)\psi(s) - \phi(s)\psi^{[1]}(b))g(s)\Delta s.$$

Substituting these values into the second condition of (6.5) yields

$$c_1(\gamma\phi(b) + \delta\phi^{[1]}(b)) + \frac{\gamma\phi(b) + \delta\phi^{[1]}(b)}{w}\int_a^b \psi(s)g(s)\Delta s = 0.$$

Again using the definition of w,

$$\gamma\phi(b) + \delta\phi^{[1]}(b) = -w \neq 0.$$

Hence

$$c_1 = -\frac{1}{w}\int_a^b \psi(s)g(s)\Delta s.$$

Thus y has the desired form, and the Green function $G(t,s)$ is symmetric. □

Remark 6.7. It can be verified that for the solution y of the non-homogeneous equation (6.4), under the non-homogeneous boundary conditions

$$\alpha y(a) - \beta y^{[1]}(a) = d_1, \quad \gamma y(b) + \delta y^{[1]}(b) = d_2,$$

the formula

$$y(t) = \frac{d_2}{w}\phi(t) - \frac{d_1}{w}\psi(t) + \int_a^b G(t,s)g(s)\Delta s$$

holds, where $G(t,s)$ is as defined in the previous theorem.

For further discussion of second-order self-adjoint dynamic equations on time scales see [2,9,27].

7. Fourth-order dynamic equations

Let $n = 2$, and consider the fourth-order dynamic expression

$$Ly(t) = (p_0 y^{\Delta\nabla})^{\nabla\Delta}(t) - (p_1 y^{\nabla})^{\Delta}(t) + p_2(t)y(t). \tag{7.1}$$

For $y \in \Omega$ we have by definition

$$
\begin{aligned}
y^{[0]} &= y,\\
y^{[1]} &= y^{\Delta},\\
y^{[2]} &= p_0 y^{\Delta\nabla},\\
y^{[3]} &= p_1 y^{\nabla} - (y^{[2]})^{\nabla},\\
y^{[4]} &= p_2 y - (y^{[3]})^{\Delta}.
\end{aligned}
$$

It follows that

$$Ly = y^{[4]}.$$

In this case, for $y, z \in \Omega$ the Lagrange bracket of y and z is

$$[y, z](t) = y^{[0]}(t)z^{[3]}(t) - y^{[3]}(t)z^{[0]}(t) + y^{[1]}(t)z^{[2]}(t) - y^{[2]}(t)z^{[1]}(t)$$

and the Lagrange identity

$$zLy - yLz = [y, z]^{\Delta}$$

holds. Using the same techniques as in previous sections, for each function $y \in \Omega$, we have the following system of relations at $t \in \mathbb{T}^*$:

$$
\begin{aligned}
(y^{[0]})^{\Delta} &= y^{[1]},\\
(y^{[1]})^{\Delta} &= -\mu_\sigma^2 \frac{p_2}{p_0^\sigma} + \mu_\sigma \frac{p_1^\sigma}{p_0^\sigma} y^{[1]} + \frac{1}{p_0^\sigma} y^{[2]} - \frac{\mu_\sigma}{p_0^\sigma} y^{[3]} + \frac{\mu_\sigma^2}{p_0^\sigma} Ly,\\
(y^{[2]})^{\Delta} &= -\mu_\sigma p_2 y^{[0]} + p_2^\sigma y^{[1]} - y^{[3]} + \mu_\sigma Ly,\\
(y^{[3]})^{\Delta} &= p_2 y^{[0]} - Ly.
\end{aligned}
$$

Thus the dynamic equation $Ly(t) = g(t)$ for $t \in \mathbb{T}^*$ where $g : \mathbb{T} \to \mathbb{R}$ is a right-dense continuous function is equivalent to the first-order system

$$\vec{y}^{\Delta}(t) = A(t)\vec{y}(t) + \vec{g}(t), \quad t \in \mathbb{T}^*,$$

where

$$
\vec{y} = \begin{bmatrix} y^{[0]} \\ y^{[1]} \\ y^{[2]} \\ y^{[3]} \end{bmatrix}, \quad
\vec{g} = \begin{bmatrix} 0 \\ \mu_\sigma^2 \dfrac{g}{p_0^\sigma} \\ \mu_\sigma g \\ -g \end{bmatrix}, \quad
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\mu_\sigma^2 \dfrac{p_2}{p_0^\sigma} & \mu_\sigma \dfrac{p_1^\sigma}{p_0^\sigma} & \dfrac{1}{p_0^\sigma} & -\dfrac{\mu_\sigma}{p_0^\sigma} \\ \mu_\sigma p_2 & p_1^\sigma & 0 & -1 \\ p_2 & 0 & 0 & 0 \end{bmatrix}.
$$

Together with the dynamic expression (7.1), take boundary conditions of the form

$$\sum_{k=1}^{4} \alpha_{jk} y^{[k-1]}(a) + \sum_{k=1}^{4} \beta_{jk} y^{[k-1]}(b) = 0, \quad 1 \leqslant j \leqslant 4. \tag{7.2}$$

These boundary conditions are self-adjoint, if and only if

$$
\begin{aligned}
0 &= y(b)z^{[3]}(b) - y^{[3]}(b)z(b) + y^{[1]}(b)z^{[2]}(b) - y^{[2]}(b)z^{[1]}(b) \\
&\quad - y(a)z^{[3]}(a) + y^{[3]}(a)z(a) - y^{[1]}(a)z^{[2]}(a) + y^{[2]}(a)z^{[1]}(a)
\end{aligned}
$$

for all $y, z \in \Omega_{[a,b)}$. It follows that by joining any one of the four types of conditions

(i) $y(a) = y^{[1]}(a) = 0$,
(ii) $y^{[1]}(a) = y^{[3]}(a) = 0$,
(iii) $y(a) = y^{[2]}(a) = 0$,
(iv) $y^{[2]}(a) = y^{[3]}(a) = 0$

with any one of the four types of conditions

(i) $y(b) = y^{[1]}(b) = 0$,
(ii) $y^{[1]}(b) = y^{[3]}(b) = 0$,
(iii) $y(b) = y^{[2]}(b) = 0$,
(iv) $y^{[2]}(b) = y^{[3]}(b) = 0$,

yields the 16 types of self-adjoint boundary conditions. The "periodic" boundary conditions

$$y(a) = y(b), \quad y^{[1]}(a) = y^{[1]}(b), \quad y^{[2]}(a) = y^{[2]}(b), \quad y^{[3]}(a) = y^{[3]}(b),$$

are also self-adjoint.

For the remainder of this section we consider the dynamic expression

$$Ly(t) = (py^{\Delta\nabla})^{\nabla\Delta}, \quad t \in [a, b).$$

Theorem 7.1. *The Green's function $G(t, s)$ of Ly with the boundary conditions*

$$y(a) = y^{[1]}(a) = y^{[2]}(b) = y^{[3]}(b) = 0$$

is given by

$$G(t,s) = \begin{cases} \int_a^t \left(\int_a^\tau \dfrac{s-x}{p(x)} \nabla x \right) \Delta\tau, & t \leqslant s, \\[2ex] \int_a^s \left(\int_a^\tau \dfrac{t-x}{p(x)} \nabla x \right) \Delta\tau, & t \geqslant s. \end{cases}$$

Remark 7.2. The boundary conditions

$$y(a) = y^{[1]}(a) = y^{[2]}(b) = y^{[3]}(b) = 0$$

can be rewritten in the form

$$y(a) = y^{\Delta}(a) = y^{\Delta\nabla}(b) = y^{\Delta\nabla^2}(b) = 0.$$

Example 7.3. Let $\mathbb{R}$, $h\mathbb{Z}$, and $\mathbb{E}$ be as defined in Example 2.2, and suppose that $hm = 1$ for some integer $m > 1$. Taking $a = 0$ and $b = 1$ with $p(t) \equiv 1$ we have the following [8]:

$$\mathbb{T} = \mathbb{R} : G(t,s) = \begin{cases} \dfrac{t^2(3s-t)}{6}, & t \leqslant s \\ \dfrac{s^2(3t-s)}{6}, & t \geqslant s, \end{cases}$$

$$\mathbb{T} = h\mathbb{Z} : G(t,s) = \begin{cases} \dfrac{t\rho(t)(3s-\sigma(t))}{6}, & t \leqslant s, \\ \dfrac{s\rho(s)(3t-\sigma(s))}{6}, & t \geqslant s, \end{cases}$$

$$\mathbb{T} = \mathbb{E} : G(t,s) = \begin{cases} \dfrac{t\rho(t)((q^2+q+1)s-\sigma(t))}{(q+1)(q^2+q+1)}, & t \leqslant s, \\ \dfrac{s\rho(s)((q^2+q+1)t-\sigma(s))}{(q+1)(q^2+q+1)}, & t \geqslant s. \end{cases}$$

We can see that in these three cases the Green function is symmetric, and that $\mathbb{R}$ is the limiting case as $h \to 0$ and $q \to 1$, respectively.

Remark 7.4. In [7] it is shown (Example 18) that in the case $\mathbb{T} = \mathbb{Z}$ the Green function of

$$Ly(t) = (y^{\Delta^2})^{\nabla^2}(t)$$

with the boundary conditions

$$y(a) = y^{\Delta}(a) = y^{\Delta^2}(b) = y^{\Delta^2\nabla}(b) = 0$$

is not symmetric. We can see that this expression for Ly is in the form (7.1) with $p(t) \equiv 1$ since in the case $\mathbb{T} = \mathbb{Z}$ the operators Δ and ∇ commute. However, these boundary conditions, in contrast to the boundary conditions

$$y(a) = y^{\Delta}(a) = y^{\Delta\nabla}(b) = y^{\Delta\nabla^2}(b) = 0,$$

are not self-adjoint. This is why the Green function turned out to be non-symmetric. Note also that if we replace in the self-adjoint boundary conditions for $\mathbb{T} = \mathbb{R}$ the real derivative with the delta or nabla derivative, the resulting boundary conditions need not be self-adjoint.

Theorem 7.5. *The Green's function of* $Ly = (py^{\Delta\nabla})^{\nabla\Delta}$ *with the self-adjoint boundary conditions*

$$y^{\Delta\nabla}(a) = y^{\Delta\nabla^2}(a) = y(b) = y^{\Delta}(b) = 0$$

is given by

$$G(t,s) = \begin{cases} \int_s^b \left(\int_\tau^b \dfrac{x-t}{p(x)} \nabla x \right) \Delta\tau, & t \leqslant s \\ \int_t^b \left(\int_\tau^b \dfrac{x-s}{p(x)} \nabla x \right) \Delta\tau, & t \geqslant s. \end{cases}$$

If $p(t) \equiv 1$ *we have*

$$\mathbb{T} = \mathbb{R}: \quad G(t,s) = \begin{cases} \dfrac{(b-s)^2(2b+s-3t)}{6}, & t \leqslant s \\ \dfrac{(b-t)^2(2b+t-3s)}{6}, & t \geqslant s. \end{cases}$$

Theorem 7.6. *The Green's function of* $Ly = (py^{\Delta\nabla})^{\nabla\Delta}$ *with the self-adjoint boundary conditions*

$$y^{[0]}(a) = y^{[2]}(a) = y^{[1]}(b) = y^{[3]}(b) = 0$$

is given by

$$G(t,s) = \begin{cases} (t-a)\left(\int_a^s \dfrac{x-a}{p(x)} \nabla x + \int_s^b \dfrac{s-a}{p(x)} \nabla x \right) - \int_a^t \int_a^\tau \dfrac{x-a}{p(x)} \nabla x \Delta\tau, & t \leqslant s \\ (s-a)\left(\int_a^t \dfrac{x-a}{p(x)} \nabla x + \int_t^b \dfrac{t-a}{p(x)} \nabla x \right) - \int_a^s \int_a^\tau \dfrac{x-a}{p(x)} \nabla x \Delta\tau, & t \geqslant s. \end{cases}$$

If $p(t) \equiv 1$, *we have*

$$\mathbb{T} = \mathbb{R}: \quad G(t,s) = \begin{cases} \dfrac{(t-a)(s-a)(2b-s-a)}{2} + \frac{(a-t)^3}{6}, & t \leqslant s \\ \dfrac{(s-a)(t-a)(2b-t-a)}{2} + \frac{(a-s)^3}{6}, & t \geqslant s. \end{cases}$$

For boundary conditions

$$y^{[1]}(a) = y^{[3]}(a) = y^{[0]}(b) = y^{[2]}(b) = 0,$$

the Green function is

$$G(t,s) = \begin{cases} (b-t) \int_s^b \int_a^\tau \dfrac{\nabla x}{p(x)} \Delta\tau - \int_s^b \int_t^\tau \dfrac{x-t}{p(x)} \nabla x \Delta\tau & : t \leqslant s \\ (b-s) \int_t^b \int_a^\tau \dfrac{\nabla x}{p(x)} \Delta\tau - \int_t^b \int_s^\tau \dfrac{x-s}{p(x)} \nabla x \Delta\tau & : t \geqslant s. \end{cases}$$

8. Self-adjoint dynamic expression (1.2)

In this section, we consider the $2n$th-order dynamic expression My defined in (1.2), in which the coefficient functions $p_i : \mathbb{T} \to \mathbb{R}$ are left-dense continuous for $0 \leqslant i \leqslant n$, and $p_0(t) \neq 0$ for all $t \in \mathbb{T}$. Properties of this expression and equations involving it are analogous to those for the expression Ly given in (1.1). Here, we restrict ourselves to rewriting the dynamic equation $My = g$ in the form of a first-order equivalent system and giving an existence and uniqueness theorem.

Denote by Ω' the linear set of all functions $y : \mathbb{T} \to \mathbb{R}$, such that the function

$$(p_i y^{\nabla^{n-i-1}\Delta})^{\Delta^{n-i-1}\nabla}$$

is defined on $\mathbb{T}^{\kappa^{n-i}}_{\kappa^{n-i}}$ and is left-dense continuous function on $\mathbb{T}^*$. For each function $y \in \Omega'$, at $t \in \mathbb{T}^*$ set

$$\begin{aligned}
y^{[k]} &= y^{\nabla^k}, \quad 0 \leqslant k \leqslant n-1, \quad y^{[0]} = y^{\nabla^0} = y,\\
y^{[n]} &= p_0 y^{\nabla^{n-1}\Delta},\\
y^{[n+k]} &= p_k y^{\nabla^{n-k-1}\Delta} - (y^{[n+k-1]})^{\Delta}, \quad 1 \leqslant k \leqslant n-1\\
y^{[2n]} &= p_n y - (y^{[2n-1]})^{\nabla}.
\end{aligned}$$

We call the functions $y^{[i]}$, $0 \leqslant i \leqslant 2n$, the quasi-derivatives of y with respect to the expression My. It follows that

$$My(t) = y^{[2n]}(t).$$

For $y, z \in \Omega'$ we define the Lagrange bracket of y and z by

$$[y, z](t) = \sum_{k=1}^{n} (y^{[k-1]}(t) z^{[2n-k]}(t) - y^{[2n-k]}(t) z^{[k-1]}(t)).$$

Then the Lagrange identity

$$z(t)My(t) - y(t)Mz(t) = [y, z]^{\nabla}(t)$$

holds. Hence

$$\int_a^b z(t)My(t)\nabla t - \int_a^b y(t)Mz(t)\nabla t = [y, z]_a^b.$$

As in the case of the expression Ly, the definitions of the quasi-derivatives may be used to show that the following set of relations hold for $y \in \Omega'$:

$$(y^{[k]})^{\nabla} = y^{\nabla^{k+1}}, \quad 0 \leqslant k \leqslant n-2,$$

$$(y^{[n-1]})^{\nabla} = \frac{1}{p_0^{\rho}} y^{[n]} + \frac{\mu_\rho}{p_0^{\rho}} \left((-\mu_\rho)^{n-1} (p_n y^{[0]} - My) + \sum_{i=0}^{n-k-1} (-\mu_\rho)^i (p_{k+i}^{\rho} y^{[n-k-i]} - y^{[n+i+1]}) \right),$$

$$(y^{[n+k-1]})^{\nabla} = (-\mu_\rho)^{n-k}(p_n y^{[0]} - My) + \sum_{i=0}^{n-k-1} (-\mu_\rho)^i (p_{k+i}^{\rho} y^{[n-k-i]} - y^{[n+i+1]}),$$
$$1 \leqslant k \leqslant n-1,$$

$$(y^{[2n-1]})^{\nabla} = p_n y^{[0]} - My.$$

Therefore the dynamic equation

$$My(t) = g(t) \quad \text{for } t \in \mathbb{T}^*,$$

where $g : \mathbb{T} \to \mathbb{R}$ is a left-dense continuous function, is equivalent to the first-order nabla dynamic system

$$\vec{y}^{\nabla} = B(t)\vec{y} + \vec{g} \quad \text{for } t \in \mathbb{T}^*,$$

where

$$\vec{y} = [y^{[0]}, y^{[1]}, \ldots, y^{[2n-1]}]^{\mathrm{T}}$$

$$\vec{g} = \left[0, \ldots, 0, \frac{(-\mu_\rho)^n}{p_0^{\rho}} g, \mu_\rho g, \ldots, -(-\mu_\rho)^{n-1} g, -g\right]^{\mathrm{T}}$$

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ \frac{-(-\mu_\rho)^n p_n}{p_0^{\rho}} & \frac{-(-\mu_\rho)^{n-1} p_{n-1}^{\rho}}{p_0^{\rho}} & \frac{-(-\mu_\rho)^{n-2} p_{n-2}^{\rho}}{p_0^{\rho}} & \frac{-(-\mu_\rho)^{n-3} p_{n-3}^{\rho}}{p_0^{\rho}} & \cdots & \frac{-\mu_\rho^2 p_2^{\rho}}{p_0^{\rho}} & \frac{\mu_\rho p_1^{\rho}}{p_0^{\rho}} \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{p_0^{\rho}} & \frac{-\mu_\rho}{p_0^{\rho}} & \frac{(-\mu_\rho)^2}{p_0^{\rho}} & \cdots & \frac{(-\mu_\rho)^{n-3}}{p_0^{\rho}} & \frac{(-\mu_\rho)^{n-2}}{p_0^{\rho}} & \frac{(-\mu_\rho)^{n-1}}{p_0^{\rho}} \end{bmatrix},$$

$$B_3 = \begin{bmatrix} (-\mu_\rho)^{n-1} p_n & (-\mu_\rho)^{n-2} p_{n-1}^\rho & (-\mu_\rho)^{n-3} p_{n-2}^\rho & (-\mu_\rho)^{n-4} p_{n-3}^\rho & \cdots & \mu_\rho p_2^\rho & p_1^\rho \\ (-\mu_\rho)^{n-2} p_n & (-\mu_\rho)^{n-3} p_{n-1}^\rho & (-\mu_\rho)^{n-4} p_{n-2}^\rho & (-\mu_\rho)^{n-5} p_{n-3}^\rho & \cdots & p_2^\rho & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (-\mu_\rho)^2 p_n & -\mu_\rho p_{n-1}^\rho & p_{n-2}^\rho & 0 & \cdots & 0 & 0 \\ -\mu_\rho p_n & p_{n-1}^\rho & 0 & 0 & \cdots & 0 & 0 \\ p_n & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$B_4 = \begin{bmatrix} -1 & \mu_\rho & \cdots & -(-\mu_\rho)^{n-4} & -(-\mu_\rho)^{n-3} & -(-\mu_\rho)^{n-2} \\ 0 & -1 & \cdots & -(-\mu_\rho)^{n-5} & -(-\mu_\rho)^{n-4} & -(-\mu_\rho)^{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \mu_\rho & -(-\mu_\rho)^2 \\ 0 & 0 & \cdots & 0 & -1 & \mu_\rho \\ 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

so that

$$B(t) = \begin{bmatrix} B_1(t) & B_2(t) \\ B_3(t) & B_4(t) \end{bmatrix}$$

is an $(2n) \times (2n)$ matrix. The proof of the following theorem follows in the same fashion as the proof of existence and uniqueness for $Ly(t) = g(t)$.

Theorem 8.1 (*Existence and Uniqueness*). *Fix $t_0 \in \mathbb{T}^*$ and let c_i, $0 \leqslant i \leqslant 2n-1$, be given real constants. Then $My(t) = g(t)$ has a unique solution $y : \mathbb{T} \to \mathbb{R}$, such that $y^{[k]}(t_0) = c_k$, $0 \leqslant k \leqslant 2n-1$.*

Remark 8.2. In the boundary-value problems for the dynamic expression $My(t)$ defined in (1.2) the variable t must be in the left-open and right-closed interval $(a, b]$ and the solutions are functions defined on $[\rho^{n-1}(a), \sigma^n(b)]$.

9. Other adjoint and self-adjoint forms

In this section we consider even-order dynamic equations with alternating nabla and delta derivatives. If $\mathbb{T} = \mathbb{R}$, see [15, Section 3.6]. For sufficiently smooth functions $u, w, p_k : \mathbb{T} \to \mathbb{R}$ for $k = 0, 1, \ldots, n$ with $p_n \neq 0$, let

$$\mathscr{L}_{2n} u = p_n u^{(\nabla\Delta)^n} + p_{n-1} u^{(\nabla\Delta)^{n-1}} + \cdots + p_1 u^{\nabla\Delta} + p_0 u$$

and

$$\mathscr{L}_{2n}^{\dagger} w = (p_n w)^{(\nabla\Delta)^n} + (p_{n-1} w)^{(\nabla\Delta)^{n-1}} + \cdots + (p_1 w)^{\nabla\Delta} + p_0 w,$$

where $(\nabla\Delta)^1=\nabla\Delta$, $(\nabla\Delta)^2=\nabla\Delta\nabla\Delta$, and so on; the results here are easily mimicked for the analogous case involving alternating delta and nabla derivatives. Next, define the Lagrange bracket for smooth functions u and w recursively as follows:

$$\{u, w\}_{2(1)} := u(p_1 w)^{\nabla} - u^{\nabla}(p_1 w),$$

$$\{u, w\}_{2(2)} := u(p_2 w)^{\nabla\Delta\nabla} - u^{\nabla}(p_2 w)^{\nabla\Delta} + u^{\nabla\Delta}(p_2 w)^{\nabla} - u^{\nabla\Delta\nabla}(p_2 w) + \{u, w\}_{2(1)}$$

and

$$\begin{aligned}\{u, w\}_{2n} := {} & u(p_n w)^{(\nabla\Delta)^{n-1}\nabla} - u^{\nabla}(p_n w)^{(\nabla\Delta)^{n-1}} + \cdots + u^{(\nabla\Delta)^{n-1}}(p_n w)^{\nabla} \\ & - u^{(\nabla\Delta)^{n-1}\nabla}(p_n w) + \{u, w\}_{2(n-1)}\end{aligned}$$

for general integers $n \geqslant 2$. Then the operator $\mathscr{L}^{\dagger}_{2n}$ is the adjoint of $\mathscr{L}_{2n}$ in the sense of the next theorem.

Theorem 9.1. (*Lagrange Identity*). *The Lagrange bracket satisfies*

$$\{u, w\}^{\Delta}_{2n} = u\mathscr{L}^{\dagger}_{2n} w - w\mathscr{L}_{2n} u.$$

In particular, $\{u, w\}_{2n} \equiv \text{constant}$ *for solutions u of $\mathscr{L}_{2n}u = 0$ and w of the adjoint equation $\mathscr{L}^{\dagger}_{2n} w = 0$.*

Proof. We proceed by mathematical induction. For $n = 1$,

$$\{u, w\}^{\Delta}_{2(1)} = u(p_1 w)^{\nabla\Delta} - u^{\nabla\Delta}(p_1 w) = u\mathscr{L}^{\dagger}_{2(1)} w - w\mathscr{L}_{2(1)} u.$$

Assuming

$$\{u, w\}^{\Delta}_{2(n-1)} = u\mathscr{L}^{\dagger}_{2(n-1)} w - w\mathscr{L}_{2(n-1)} u$$

and using the product rule for delta derivatives, it follows that:

$$\begin{aligned}\{u, w\}^{\Delta}_{2n} &= u(p_n w)^{(\nabla\Delta)^n} + u^{\Delta}(p_n w)^{(\nabla\Delta)^{n-1}\Delta} - u^{\Delta}(p_n w)^{(\nabla\Delta)^{n-1}\Delta} - u^{\nabla\Delta}(p_n w)^{(\nabla\Delta)^{n-1}} \\ &\qquad + \cdots + u^{(\nabla\Delta)^{n-1}}(p_n w)^{\nabla\Delta} + u^{(\nabla\Delta)^{n-1}\Delta}(p_n w)^{\Delta} \\ &\qquad - u^{(\nabla\Delta)^{n-1}\Delta}(p_n w)^{\Delta} - u^{(\nabla\Delta)^n}(p_n w) + \{u, w\}^{\Delta}_{2(n-1)} \\ &= u(p_n w)^{(\nabla\Delta)^n} - u^{(\nabla\Delta)^n}(p_n w) + u\mathscr{L}^{\dagger}_{2(n-1)} w - w\mathscr{L}_{2(n-1)} u \\ &= u\mathscr{L}^{\dagger}_{2n} w - w\mathscr{L}_{2n} u. \qquad \square\end{aligned}$$

To find the self-adjoint form, define $\mathscr{L}_{2n}$ recursively as follows:

$$\mathscr{L}_{2(1)}u = (p_1 u^{\nabla})^{\Delta} + p_0 u,$$

$$\mathscr{L}_{2(2)}u = (p_2 u^{\nabla\Delta})^{\nabla\Delta} + \mathscr{L}_{2(1)}u,$$

$$\mathscr{L}_{2(3)}u = (p_3 u^{\nabla\Delta\nabla})^{\Delta\nabla\Delta} + \mathscr{L}_{2(2)}u,$$

$$\mathscr{L}_{2n}u = \begin{cases} (p_n u^{(\nabla\Delta)^{n/2}})^{(\nabla\Delta)^{n/2}} + \mathscr{L}_{2(n-1)}u : n \text{ even}, \\ (p_n u^{(\nabla\Delta)^{(n-1)/2}\nabla})^{(\Delta\nabla)^{(n-1)/2}\Delta} + \mathscr{L}_{2(n-1)}u : n \text{ odd}. \end{cases} \tag{9.1}$$

Then the Lagrange bracket is defined via

$$\{u, w\}_{2(1)} = u(p_1 w^{\nabla}) - w(p_1 u^{\nabla}),$$

$$\{u, w\}_{2(2)} = \{u, w\}_{2(1)} + u(p_2 w^{\nabla\Delta})^{\nabla} - u^{\nabla}(p_2 w^{\nabla\Delta}) + w^{\nabla}(p_2 u^{\nabla\Delta}) - w(p_2 u^{\nabla\Delta})^{\nabla}$$

and

$$\{u, w\}_{2n} = \{u, w\}_{2(n-1)} + u(p_n w^{(\nabla\Delta)^{n/2}})^{(\nabla\Delta)^{(n-2)/2}\nabla} - u^{\nabla}(p_n w^{(\nabla\Delta)^{n/2}})^{(\nabla\Delta)^{(n-1)/2}}$$
$$+ \cdots + w^{\nabla}(p_n u^{(\nabla\Delta)^{n/2}})^{(\nabla\Delta)^{(n-1)/2}} - w(p_n u^{(\nabla\Delta)^{n/2}})^{(\nabla\Delta)^{(n-2)/2}\nabla}$$

if n is even,

$$\{u, w\}_{2n} = u(p_n w^{(\nabla\Delta)^{(n-1)/2}\nabla})^{(\Delta\nabla)^{(n-1)/2}} - u^{\nabla}(p_n w^{(\nabla\Delta)^{(n-1)/2\nabla}})^{(\Delta\nabla)^{(n-3)/2}\Delta} + \cdots$$
$$+ w^{\nabla}(p_n u^{(\nabla\Delta)^{(n-1)/2}\nabla})^{(\Delta\nabla)^{(n-3)/2}\Delta} - w(p_n u^{(\nabla\Delta)^{(n-1)/2}\nabla})^{(\Delta\nabla)^{(n-1)/2}} + \{u, w\}_{2(n-1)}$$

if n is odd.

Theorem 9.2. (*Self-adjoint form*). *The linear operator* $\mathscr{L}_{2n}$ *in* (9.1) *is formally self-adjoint, in other words*

$$\langle u, \mathscr{L}_{2n}w\rangle = \langle w, \mathscr{L}_{2n}u\rangle$$

if and only if the Lagrange bracket defined above satisfies

$$\{u, w\}_{2n}(b) - \{u, w\}_{2n}(a) = 0.$$

Proof. As in the proof of Theorem 9.1, the Lagrange bracket of two smooth functions u and w satisfies

$$\{u, w\}_{2n}^{\Delta} = u\mathscr{L}_{2n}w - w\mathscr{L}_{2n}u.$$

Using the inner product given earlier by

$$\langle x, y\rangle = \int_a^b x(t)y(t)\Delta t,$$

the result follows. □

The following discussion on linear dynamic Hamiltonian systems can be given in terms of general n (even or odd); we illustrate the cases $n = 4, 5$. Let u be a solution of $\mathscr{L}_{2(4)}u = 0$ in (9.1). Introduce the vector functions

$$y = \begin{bmatrix} (p_4 u^{\nabla\Delta\nabla\Delta})^{\nabla\Delta\nabla} + (p_3 u^{\nabla\Delta\nabla})^{\Delta\nabla} + (p_2 u^{\nabla\Delta})^{\nabla} + p_1 u^{\nabla} \\ (p_4 u^{\nabla\Delta\nabla\Delta})^{\nabla} + p_3 u^{\nabla\Delta\nabla} \\ u^{\nabla\Delta\nabla} \\ u^{\nabla} \end{bmatrix}$$

and

$$z = \begin{bmatrix} u \\ u^{\nabla\Delta} \\ p_4 u^{\nabla\Delta\nabla\Delta} \\ (p_4 u^{\nabla\Delta\nabla\Delta})^{\nabla\Delta} + (p_3 u^{\nabla\Delta\nabla})^{\Delta} + p_2 u^{\nabla\Delta} \end{bmatrix}$$

and define the 4×4 matrices B and C as follows:

$$B = \begin{bmatrix} -p_0 & 0 & 0 & 0 \\ 0 & -p_2 & 0 & 1 \\ 0 & 0 & 1/p_4 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -p_3^{\sigma} & 0 \\ 1 & 0 & 0 & -p_1^{\sigma} \end{bmatrix}.$$

Clearly B and C are symmetric, and since u is a solution of $\mathscr{L}_{2(4)}u = 0$, it is easy to check that y, z is a solution pair to the linear dynamic Hamiltonian system

$$y^{\Delta} = Bz, \qquad z^{\Delta} = Cy^{\sigma}.$$

Similarly, let u be a solution of $\mathscr{L}_{2(5)}u = 0$, and set the vector functions

$$y = \begin{bmatrix} (p_5 u^{\nabla\Delta\nabla\Delta\nabla})^{\Delta\nabla\Delta\nabla} + (p_4 u^{\nabla\Delta\nabla\Delta})^{\nabla\Delta\nabla} + (p_3 u^{\nabla\Delta\nabla})^{\Delta\nabla} + (p_2 u^{\nabla\Delta})^{\nabla} + p_1 u^{\nabla} \\ (p_5 u^{\nabla\Delta\nabla\Delta\nabla})^{\Delta\nabla} + (p_4 u^{\nabla\Delta\nabla\Delta})^{\nabla} + p_3 u^{\nabla\Delta\nabla} \\ p_5 u^{\nabla\Delta\nabla\Delta\nabla} \\ u^{\nabla\Delta\nabla} \\ u^{\nabla} \end{bmatrix}$$

and

$$z = \begin{bmatrix} u \\ u^{\nabla\Delta} \\ u^{\nabla\Delta\nabla\Delta} \\ (p_5 u^{\nabla\Delta\nabla\Delta\nabla})^{\Delta} + p_4 u^{\nabla\Delta\nabla\Delta} \\ (p_5 u^{\nabla\Delta\nabla\Delta\nabla})^{\Delta\nabla\Delta} + (p_4 u^{\nabla\Delta\nabla\Delta})^{\nabla\Delta} + (p_3 u^{\nabla\Delta\nabla})^{\Delta} + p_2 u^{\nabla\Delta} \end{bmatrix}.$$

Letting

$$B = \begin{bmatrix} -p_0 & 0 & 0 & 0 & 0 \\ 0 & -p_2 & 0 & 0 & 1 \\ 0 & 0 & -p_4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1/p_5^{\sigma} & 0 & 0 \\ 0 & 1 & 0 & -p_3^{\sigma} & 0 \\ 1 & 0 & 0 & 0 & -p_1^{\sigma} \end{bmatrix},$$

y, z is again a solution pair to the linear dynamic Hamiltonian system

$$y^{\Delta} = Bz, \quad z^{\Delta} = Cy^{\sigma}.$$

The disconjugacy condition either way is given in terms of

$$\begin{aligned} \int_a^b [y^{\sigma}(t) \cdot (C(t)y^{\sigma}(t)) + z(t) \cdot (B(t)z(t))]\Delta t &= \int_a^b [y^{\sigma}(t) \cdot z^{\Delta}(t) + z(t) \cdot y^{\Delta}(t)]\Delta t \\ &= \int_a^b [y(t) \cdot z(t)]^{\Delta}\Delta t = y(t) \cdot z(t)|_a^b. \end{aligned}$$

What if the delta and nabla derivatives are stacked? Again for any positive integer n and sufficiently smooth functions $u, w, p_k : \mathbb{T} \to \mathbb{R}$ for $k = 0, 1, \ldots, n$ with $p_n \neq 0$, define the stacked differential operators

$$L_n u = \sum_{k=0}^{n} p_k^{\sigma} u^{\Delta^k}, \qquad L_n^{\dagger} w = \sum_{k=0}^{n} (-1)^k \, (p_k w)^{\nabla^k}.$$

As usual we introduce a Lagrange bracket for smooth functions u and w

$$\{u, w\}_n(t) := \sum_{k=0}^{n-1} u^{\Delta^k}(t) \sum_{j=0}^{n-1-k} (-1)^{j+1} (p_{k+j+1} w)^{\nabla^j}(t),$$

so that $L_n^{\dagger}$ is the adjoint of L_n in the sense of the next theorem.

Theorem 9.3 (*Lagrange Identity*). *The Lagrange bracket satisfies*

$$\{u, w\}_n^{\Delta}(t) = u(t)(L_n^{\dagger} w)^{\sigma}(t) - w^{\sigma}(t)(L_n u)(t).$$

In particular, $\{u, w\}_n \equiv \text{constant}$ *for solutions* u *of* $L_n u = 0$ *and* w *of the adjoint equation* $L_n^{\dagger} w = 0$.

Proof. For smooth functions u, w we have

$$\{u, w\}_n^\Delta = \sum_{k=0}^{n-1} u^{\Delta^{k+1}} \sum_{j=0}^{n-1-k} (-1)^{j+1} (p_{k+j+1} w)^{\nabla^j \sigma} + \sum_{k=0}^{n-1} u^{\Delta^k} \sum_{j=0}^{n-1-k} (-1)^{j+1} (p_{k+j+1} w)^{\nabla^j \Delta},$$

where, as in the proof of Lemma 3.4, we have employed the product rule for delta derivatives. Next break off the $j = 0$ term in the first expression and the $k = 0$ term in the second to obtain

$$\begin{aligned}\{u, w\}_n^\Delta = {} & \sum_{k=0}^{n-2} u^{\Delta^{k+1}} \sum_{j=1}^{n-1-k} (-1)^{j+1} (p_{k+j+1} w)^{\nabla^{j-1} \Delta} - \sum_{k=0}^{n-1} u^{\Delta^{k+1}} (p_{k+1} w)^\sigma \\ & + \sum_{k=1}^{n-1} u^{\Delta^k} \sum_{j=0}^{n-1-k} (-1)^{j+1} (p_{k+j+1} w)^{\nabla^j \Delta} + u \sum_{j=0} (-1)^{j+1} (p_{j+1} w)^{\nabla^{j+1} \sigma};\end{aligned}$$

after reindexing the sums, we have

$$\begin{aligned}\{u, w\}_n^\Delta &= u \sum_{j=1}^{n} (-1)^j (p_j w)^{\nabla^j \sigma} - w^\sigma \sum_{k=1}^{n} p_k^\sigma u^{\Delta^k} = u((L_n^\dagger w)^\sigma - p_0^\sigma w^\sigma) - w^\sigma (L_n u - p_0^\sigma u) \\ &= u(L_n^\dagger w)^\sigma - w^\sigma (L_n u). \qquad \square\end{aligned}$$

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